Periodic Optimal Cruise of an Atmospheric Vehicle

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Since the steady-state cruise path of an idealized point mass model of an atmospheric vehicle operating in the hypersonic flight regime is dynamically not fuel minimizing, closed periodic paths are numerically determined. By application of second-order conditions for local optimality, a periodic extremal path for a flat Earth is shown to be locally minimizing and produces an improvement in fuel usage of 4.2% over the steady-state cruise path. Application of these second variational conditions to extremal paths for the spherical Earth failed. Nevertheless, these paths produce improved fuel performance over the associated steady-state cruise path.

I. Introduction

FUEL efficient cruise trajectories for aircraft have been a subject of continuous theoretical interest and are becoming one of practical interest as well. The analysis of aircraft trajectories for fuel minimization was first performed in a reduced state space.[1-3] By neglecting the altitude and flight path angle dynamics, the equations of motion of a point mass representation of the aircraft motion reduces to the energy-state approximation where energy and fuel mass are the state variables, thrust and velocity (or altitude) are considered the control variables, and range is the independent variable. The first-order dynamics or rates are represented by the rate of change in the energy and fuel with respect to a change in range. The hodograph[3] is formed by determining the boundary of reachable rates for admisible values of the control variables at a given energy value (the rates are independent of the fuel). The steady-state cruise fuel performance is given by the value of the fuel mass rate where the hodograph crosses the zero energy rate axis. If the hodograph is not convex so that a straight line tangent to two points on the hodograph (called the convex hull[3]) crosses the zero energy rate axis at a smaller value of fuel mass rate than does the hodograph, then the control variables at the points of tangency are used to form a chattering control sequence which theoretically will improve fuel performance over the steady-state cruise path. This chattering sequence, first discussed in Ref. 1, is an unrealizable infinite frequency control sequence between two thrust levels and two altitude and velocity points on the energy manifold where the aircraft is aerodynamic or propulsion efficient. This chattering cruise is also referred to as the *relaxed* steady state cruise in Ref. 2.

Since velocity and altitude chattering is unrealizable, altitude has been added as a state variable in Ref. 4 and thrust and flight path angle are considered the control variables. In Ref. 4 the small angle approximation applied to the flight path angle results in flight path angle and thrust appearing linearly in the aircraft dynamic model and cost function. These control variables which lie interior to their admissible control sets along the extremizing steady-state cruise path form what is called a doubly singular arc in the calculus of variations.[5,6] By applications of the matrix generalized Legendre-Clebsch condition due to Robbins,[7] it is demonstrated in Ref. 8 that the steady-state cruise path is not minimizing.

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The steady-state cruise arc using the aircraft model assumed in Ref. 4, extended to the full point mass model by including the flight path angle as a state variable and angle of attack as a control variable, is shown in Ref. 9 to satisfy the first-order necessary conditions and the generalized Legendre-Clebsch condition applied to the thrust. However, this does not mean that steady-state cruise is a locally minimizing path in the calculus of variations. A second variational test must be applied. Since steady-state cruise is time or range invariant and is assumed to go on forever, the second variational accessory problem reduces to that of an infinite time, quadratic cost problem subject to a time or range invariant linear dynamical system. By using Parseval's rule, the second variational accessory problem reduces to ensuring the positive definiteness of a frequency dependent matrix kernal.[10,11] This frequency test fails[12,13] when applied to many aircraft models including that of Ref. 6, and, therefore, steady-state cruise obtained from the full point mass aircraft model is often not a minimizing path.

Since the steady-state cruise path is not minimizing, the objective is to obtain the periodic cruise paths that are locally minimizing and determine their properties. The particular atmospheric vehicle model used was suggested in Ref. 12. For this problem the optimal steady-state cruise path lies at any point along a line in altitude-velocity space determined by an optimum dynamic pressure value. Furthermore, for the reduced energy-state approximation, chattering steady-state cruise does not exist. In Ref. 12 the second variational frequency test fails for this model if the velocity is large enough. This implies that high-velocity steady-state cruise paths are not dynamically locally optimizing even thought they satisfy all the first-order necessary conditions. This infers that there are two underlying mechanisms for producing periodic paths. The first mechanism is the mismatch in the regions of velocity and altitude where the aircraft is aerodynamic and propulsion efficient. This is the mechanism behind the chattering cruise. However, since this mechanism is lacking in this model, there is a potential and kinetic energy interchange which is optimal for fuel performance. The need for substantial kinetic energy seems to be the reason for the velocity threshold found in Ref. 12.

By using the atmospheric model of Ref. 12, periodic cruise paths are numerically obtained. To determine if these extremal periodic cruise paths are locally minimizing, the sufficiency conditions of Ref. 14, based upon the second variation, are applied. The general problem formulation, the first-order necessary conditions, and the sufficiency condition of Ref. 14 are stated in Sec. II. The sufficiency condition is extended to include the optimality condition of the thrust switch points. In Sec. III the formulation of the periodic fuel optimization problem for the atmospheric vehicle model given in Ref. 12 is presented. To determine the extremal periodic cruise path, a shooting method reported in Ref. 15 is discussed in Sec. IV. This method takes into account certain singularities that exist for periodic Hamiltonian systems. In Sec. V the numerical results of applying this optimization algorithm are displayed. The most important result is that one periodic extremal path for the flat Earth approximation is found which satisfies the conditions of Ref. 14. The fuel performance improvement of the periodic cruise over the steady-state cruise is 4.2%. Results are also given for the spherical Earth, but no locally minimizing path was found which satisfied the conditions in Ref. 14. Nevertheless, the periodic extremal cruise paths found did improve fuel performance over their associated steady-state cruise paths by as much as 4.5%. Conclusions are given in Sec. VI.

II. Problem Formulation and Conditions for Optimality

In this section the optimal periodic control problem is formulated and first- and second-order conditions for optimality are stated. These conditions of optimality are derived in Ref. 14. Those aspects of the theory which are peculiar to periodic extremal cruise paths are emphasized below since they are used to structure the numerical shooting optimization scheme.

Problem Formulation for Optimal Periodic Control

The optimal periodic control problem consists of minimizing the performance criterion

$$J\left(u(\cdot),x_o,r_p,r_i\right)=\frac{1}{r_p}\int_0^{r_p} L[x(t),u(r),r_i]\,\mathrm{d}r \qquad (1)$$

with respect to the period $r_p \epsilon T \triangleq (0,\infty)$, the p-vector control function $u(\cdot)\epsilon U$ where U is the set of piecewise continuous control functions, the initial condition of the state variables $x(0)\triangleq x_0 \epsilon \mathbb{R}^n$, and the control parameters $r_i \epsilon [0,r_p]$, $i=1,2,\ldots$, subject to the dynamic system

$$\dot{x}(t)=f[x(r),u(r),r_i] \qquad (2)$$

and the periodic boundary conditions

$$x(0)=x(r_p) \qquad (3)$$

Since time is not important in our formulation of the cruise problem, the range r is used as the independent variable and $(\dot{\ })$ denotes d( )/dr. It is assumed that f and L and their derivatives up to second-order are continuous with respect to all their arguments. Furthermore, it is assumed that a periodic optimal path exists for this problem with nonzero period so that the theory of Ref. 14 is applicable. For steady-state or equilibrium solutions the second variational theory of Refs. 10 and 11 applies.

The parameter r_i, $i=1,2,\ldots$, which will denote switch range in the thrust profile, was not included in the conditions of optimality given in Ref. 14 although its inclusion is a straightforward extension. In the numerical optimization scheme, the thrust profile is assumed bang-bang. No assumption is made on the number of switches. However, if the number of switches over some subarc becomes large, indicating a chattering subarc, then the presence of a singular arc[5,6] where the thrust is interior to its admissible control set could be hypothesized. Fortunately, for the numerical periodic paths determined here, singular arcs did not occur. Therefore, second order optimality is obtained simply by considering that the switch ranges are control parameters.[16,17]

First-Order Necessary Conditions

The first-order necessary conditions are determined by adjoining the dynamic constraints, Eq. (2), to the performance index, Eq. (1), by an n-vector Lagrange multiplier $\lambda(r)$ as well as the boundary conditions, Eq. (3), and taking the first variation. Define the variational Hamiltonian as

$$H(x,u,\lambda,r_i)=L(x,u,r_i)+\lambda^T f(x,u,r_i) \qquad (4)$$

The stationarity of the augmented cost criterion implies the first-order necessary conditions, with $(\ )_x$ denoting $\partial(\cdot)/\partial x$,

$$\dot{x}=H_\lambda^T, \qquad \dot{\lambda}=-H_x^T, \qquad 0=H_u \qquad (5)$$

with boundary conditions

$$x(0)=x(r_p),\ \lambda(0)=\lambda(r_p),\ H(r_p)-J[u(\cdot),x_0,r_p,r_i]=0 \qquad (6)$$

and at the switch range

$$J_{r_i}=H(x,u,\lambda,r_i^-)-H(x,u,\lambda,r_i^+)=0 \qquad (7)$$

where x and λ are continuous at r_i, and r_i^+ (r_i^-) denotes the functions L and f, evaluated just after (prior to) a discontinuity in these functions. The transversality condition $H-J=0$ in Eq. (6) is due to division of the performance criterion by r_p. The corner condition, Eq. (7), is found in Refs. 16 and 17.

For convenience in presenting the second variational conditions and some of the special aspects of this theory applied to periodic paths, let the state and Lagrange multiplier space be defined as

$$y(t)^T \triangleq [x(r),\lambda(r)]^T \qquad (8)$$

By using the implicit function theorem assuming that H_{uu} and $J_{r_ir_i}$ are positive definite, $H_u=0$ and $J_{r_i}=0$ imply that $u=\bar{u}(x,\lambda)$ and $r_i=\bar{r}_i(x,\lambda)$. Therefore, the variational equations in (6) can be written as

$$\dot{y}=KH_y^T[y,\bar{u}(y),\bar{r}_i(y)]; \qquad y(0)=y(r_p) \qquad (9)$$

where K is the $2n\times 2n$ fundamental symplectic matrix

$$K\triangleq\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \qquad (10)$$

A closed periodic path which satisfies Eq. (9) is called here an *orbit*. Note that the transversality condition in Eq. (6) is generally not satified. Define the monodromy matrix as the transition matrix $\Phi(r,0)$ evaluated at $r=r_p$ where the transition matrix is propagated as

$$\dot{\Phi}(r,0)=KH_{yy}(r)\Phi(r,0)\ ;\ \Phi(0,0)=I \qquad (11)$$

where KH_{yy} is evaluated along an orbit. Special care must be taken in calculating $\Phi(r,0)$ at points of discontinuity r_i. For the cruise problem this is explicitly done in the section covering calculation of the transition matrix across thrust switch points.

Second Variational Sufficient Conditions for Weak Local Optimality of Periodic Processes

The second variational conditions given in Ref. 14 are stated in this section. Let the extremal control function, the initial state vector, the period, and the switch ranges be denoted as $(u^\circ(\cdot),\ x_o^\circ,r_p^\circ,r_i^\circ)$. For the periodic control problem described in the section relating to problem formulation for optimal periodic control, $(u^\circ(\cdot)\ x_o^\circ,r_p^\circ,r_i^\circ)$ forms a weak local minimum if

i) the first-order necessary conditions (5), (6), and (7) are satisfied and $H_{uu}>0$,

ii) there exists a real valued, bounded, periodic symmetric matrix solution to the Riccati differential equation

$$\dot{P} = -PA - A^T P + PBP - C; \qquad P(0) = P(r_p^\circ) \tag{12}$$

Here the partioning into $n \times n$ blocks of the matrix KH_{yy} is defined as

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \triangleq KH_{yy} \tag{13}$$

where KH_{yy} is evaluated along the extremal orbit,

iii) there are no eigenvalues of the monodromy matrix $\Phi(r_p^\circ, 0)$ on the unit circle except for the two coupled unity eigenvalues associated with the eigenvector $\dot{y}(0)$ where $\partial r_p(H)/\partial H \neq 0$ ensures this coupling,

iv) the eigenvalues off the unit circle are distinct, and

v) $$J_{r_i r_i} > 0 \tag{14}$$

The two unity eigenvalues are characteristic of orbits obtained from Hamiltonian systems.[14,18,19] The velocity vector $\dot{y}(0) = \dot{y}(r_p^\circ)$ is an eigenvector of $\Phi(r_p^\circ, 0)$ associated with the unity eigenvalue since $\Phi(r,0)$ is the solution matrix associated with $\ddot{y} = KH_{yy}\dot{y}$.[14,18,19] Since $\Phi(r_p^\circ, 0)$ is a symplectic matrix, i.e., n eigenvalues of $\Phi(r_p^\circ, 0)$ are equal to the reciprocals of the remaining eigenvalues, there is a second unit eigenvalue. These two unity eigenvalues are coupled in the same Jordan box[14,18,19] if $\partial r_p(H)/\partial H \neq 0$, i.e., a change in the Hamiltonian implies a nonzero change in the period. Since the unity eigenvalues are coupled, then there is a generalized eigenvector associated with $\dot{y}(0)$. The direction of this generalized eigenvector indicates the change in $y(0)$ that can be made to find a neighboring orbit. This generalized eigenvector, called here y_H, is used in the numerical optimization routine to establish a one-dimensional family of orbits.

The eigenvalue restriction in condition (iii) is required because a necessary condition for the existence of a real-valued periodic solution to the Riccati equation is that there be no distinct eigenvalues of the monodromy matrix on the unit circle.[14] In the numerical algorithm the eigenvalues of the monodromy matrix are monitored. Furthermore, it is conjectured that if condition (iii) is weakened to require that no distinct eigenvalues lie on the unit circle and $\partial r_p/\partial H$ can be zero at points, then this condition is necessary as well.

If all the nonunity eigenvalues of the monodromy matrix are off the unit circle, then a real-valued, bounded, periodic solution to Eq. (11) may exist. As is well known,[20] the solution to the Riccati equation can be determined in terms of the $n \times n$ block partitioned matrices of the transition matrix $\Phi(r,0)$ as

$$P(r) = [\Phi_{21}(r,0) + \Phi_{22}(r,0)P(0)]\,[\Phi_{11}(r,0) + \Phi_{12}(r,0)P(0)]^{-1} \tag{15}$$

By evaluating Eq. (15) at $r = r_p^\circ$ and using the periodic requirement that $P(0) = P(r_p) \triangleq \boldsymbol{P}$, then Eq. (15) can be manipulated into the nonstandard algebraic Riccati equation

$$\boldsymbol{P}\Phi_{12}(r_p^\circ, 0)\boldsymbol{P} + \boldsymbol{P}\Phi_{11}(r_p^\circ, 0) - \Phi_{22}(r_p^\circ, 0)\boldsymbol{P} - \Phi_{21}(r_p^\circ, 0) = 0 \tag{16}$$

Satisfaction of Eq. (16) does not mean $P(r)$ is bounded for all $r \epsilon [0, r_p^\circ]$ and must be checked for all r in that interval.

Condition (iv) appears to be satisfied generically. It is used in Ref. 14 to ensure that the second variation is strongly positive, which means the second variation dominates the higher-order terms in the Taylor expansion of the cost criterion. Finally condition (v) will be explicitly evaluated for the periodic cruise problem using the results from Ref. 16.

III. Formulation of the Periodic Optimization Problem for an Atmospheric Vehicle

The atmospheric vehicle model to be described, in which all the numerical results to be given lie in the hypersonic region where Mach number is greater than 2, is chosen mostly because of its simplicity than for its realism. Nevertheless, it is a point mass model which illustrates a principle in the fuel optimal periodic motion of atmospheric vehicles.

The idealized equations of motion of a point mass over a spherical earth are in scaled variables

$$\bar{h}' = \tan\gamma \tag{17}$$

$$M' = gF(T - D - W\sin\gamma)/Ma^2 W\cos\gamma \tag{18}$$

$$\gamma' = gF(L - W\cos\gamma)/M^2a^2 W\cos\gamma + 1/(\bar{R}_o + \bar{h}) \tag{19}$$

$$\bar{f}' = \sigma T/Ma\cos\gamma \tag{20}$$

where the scaled variables are (F is the scale factor)

$$\bar{r} \triangleq r/F, \qquad \bar{h} \triangleq h/F, \qquad \bar{f} \triangleq f/F \tag{21}$$

where $(\;)'$ denotes $d(\;)/d\bar{r}$, r is the range, h is altitude, γ is the flight path angle, M is Mach number, a is the speed of sound, g is the gravitational acceleration, $\bar{R}_o \triangleq R_o/F$, R_o is the radius of the Earth, W is the vehicle weight which is assumed constant, f is the fuel mass used, L is the lift force, D is the drag force, σ is the thrust specific fuel consumption, and T is the thrust force bounded, in general, by the inequaity constraint

$$T_{\min}(M,h) \le T \le T_{\max}(M,h) \tag{22}$$

It is assumed that the thrust lies on its bounds except at a finite number of switch ranges. The determination of the switch ranges are given in the section covering first-order necessary conditions for periodic cruise. Note that since W is assumed constant, the equations of motion are invariant with respect to change in time or range. By using range as the independent variable, the number of equations of motion are reduced by one. The scaling is used to improve numerical accuracy by having $(\bar{r}, \bar{h}, M, \gamma)$ about the same order of magnitude.

The following idealizations are assumed to enhance the model simplicity and underscore the character of the numerical solution. The lift and drag force models are

$$L = C_L qS \qquad D = C_D qS \tag{23}$$

where

$$q = \tfrac{1}{2}\rho(Ma)^2, \quad C_D = C_{D_0} + KC_L^2 \tag{24}$$

where C_L is the lift coefficient used here as a control variable, q is the dynamic pressure, S is the wing span squared, C_{D_0} is the zero-lift drag coefficient, K is the induced drag factor, and ρ is the atmospheric density assumed to be exponential of the form

$$\rho = \rho_o e^{-\beta h} \tag{25}$$

where the parameters ρ_o and β are chosen to fit the atmospheric model of the stratosphere where the isothermal assumption is reasonable and the speed of sound is a constant. The parameters C_{D_0} and K are also assumed constant, which is not too unrealistic for hypersonic vehicles.[21]

Although the performance of supersonic combustion ramjets (scramjet) is somewhat speculative,[21] our engine model is far more ideal. The thrust specific fuel consumption is assumed to be

$$\sigma = \nu M \tag{26}$$

where ν is a constant. For high enough M this seems to be approximately the correct form for a scramjet.[21] Our most serious assumption is that $T_{min}=0$ and that T_{max} is a constant independent of m and h. Furthermore, the time to get from T_{min} to T_{max} and vice versa is assumed neglectable. Nevertheless, it is expected that our approximations are mostly conservative and that more precise modeling will lead to even better relative fuel performance, especially in the hypersonic region.

The periodic optimization problem is to find the control history $C_L(\cdot)$, the initial states $(\bar{h}(0), M(0), \gamma(0))$, the period $\bar{r}_p$, and the switch ranges of thrust discontinuities $\bar{r}_i$ (the number of switches is unspecified) which minimize the fuel to range ratio

$$J=\frac{1}{\bar{r}_p}\int_0^{\bar{r}_p}\frac{\nu T}{a\cos\gamma}\mathrm{d}\bar{r} \tag{27}$$

subject to the dynamic Eqs. (17-19) and the boundary conditions

$$\bar{h}(0)=\bar{h}(\bar{r}_p),\qquad M(0)=M(\bar{r}_p),\qquad \gamma(0)=\gamma(\bar{r}_p) \tag{28}$$

The choice of switch ranges as control parameters rather than thrust as a control function is discussed below.

First-Order Necessary Conditions for Periodic Cruise

If the states and control are identifies as $x\triangleq[\bar{h},M,\gamma]^T$ and $u(\cdot)\triangleq C_L(\cdot)$, then the first-order necessary conditions given in Sec. II can be explicitly written for the cruise problem. Unless specifially defined, subscripted variables denote partial differentiation with respect to that variable. First, the Hamiltonian is defined as

$$H\triangleq \nu T/a\cos\gamma+\lambda_{\bar{h}}\tan\gamma+\lambda_M gF(T-D-W\sin\gamma)/Ma^2W\cos\gamma$$
$$+\lambda_\gamma gF(L-W\cos\gamma)/M^2a^2W\cos\gamma+\lambda_\gamma/(\bar{R}_o+\bar{h}) \tag{29}$$

where $\lambda\triangleq[\lambda_{\bar{h}},\lambda_M,\lambda_\gamma]^T$. These Lagrange multipliers are propagated using Eq. (5) as

$$\lambda_{\bar{h}}'=\lambda_M D_{\bar{h}}Fg/WMa^2\cos\gamma-\lambda_\gamma[gFL_{\bar{h}}/WM^2a^2\cos\gamma$$
$$-1/(\bar{R}_o+\bar{h})^2] \tag{30}$$

$$\lambda_M'=[\lambda_M gF(T-D-W\sin\gamma)+\lambda_M gFD_M M$$
$$+2\lambda_\gamma gF(L-W\cos\gamma)/M-\lambda_\gamma gFL_M]/WM^2a^2\cos\gamma \tag{31}$$

$$\lambda_\gamma'=-\nu T\sin\gamma/a\cos^2\gamma-\lambda_{\bar{h}}/\cos^2\gamma-\lambda_M gF\{(T-D)\sin\gamma$$
$$-W\}/WMa^2\cos\gamma-\lambda_\gamma LgF\sin\gamma/WM^2a^2\cos^2\gamma \tag{32}$$

with boundary conditions

$$\lambda_{\bar{h}}(0)=\lambda_{\bar{h}}(r_p),\quad \lambda_M(0)=\lambda_M(r_p),\quad \lambda_\gamma(0)=\lambda_\gamma(r_p) \tag{33}$$

The optimality condition $H_u=0$ associated with C_L is

$$H_{C_L}=[-\lambda_M gD_{C_L}M+\lambda_\gamma gL_{C_L}]/WM^2a^2\cos\gamma=0 \tag{34}$$

With the choice of drag polar given in Eq. (24), $H_{C_LC_L}>0$ if $\lambda_M<0$.

By considering the thrust as a control function, the switch ranges can be determined by application of the Pontryagin minimum principle.

$$T=T_{max}\quad \text{if}\quad H_T<0$$
$$T=0\quad \text{if}\quad H_T>0 \tag{35}$$

and if $H_T=0$, the thrust switches bounds. These conditions determine both the switch ranges and the minimizing thrust bound. The condition $H_T=0$ is equivalent to condition (7) since the states and the Lagrange multipliers are assumed to be continuous. Therefore, for the cruise problem $J_{r_i}=H_T\Delta T=0$ where $\Delta T\triangleq T(r_i^-)-T(r_i^+)$. The importance of introducing the switch ranges as control parameters is that local optimality can be established by the second variational condition (v) as long as no intermediate thrusting arcs or singular arcs are present. A singular arc means that the thrust is not on its boundary but takes on values interior to its admissible control set, thereby requiring that $H_T=0$ for a finite interval of range. The arc is called singular because H_{TT} is identically zero, implying that the thrust values cannot be obtained from $H_T=0$. The assumption made in developing the numerical optimation program is that singular arcs will not occur although the occurrence of a singular arc is indicated by an interval of high-frequency switching from one thrust bound to the other. This phenomenon is permitted in our program since the number of switches is not limited. Fortunately, this pathology did not occur.

The transversality condition associated with the optimality of the range period is given in Eq. (6). As discussed in the next section, satisfaction of this condition in the numerical optimization scheme occurs last.

IV. A Shooting Method for the Numerical Solution of Optimal Periodic Cruise Paths

A shooting type numerical optimization technique is developed by taking into account the particular characteristics of the periodic optimal control problem. Starting from an arbitrary set of initial conditions $y(0)$, it is required that the extremal path be closed to form an orbit which most likely does not satisfy the transversality condition (6). Starting with this orbit, a family of orbits is followed toward satisfaction of (6), thereby determining the desired free period orbit. Along this path $H_{uu}>0$ is checked. Finally, the remaining four conditions of the section concerning second variational sufficient conditions for weak local optimality of periodic process are applied to the free period orbit to determine if it is locally minimizing. In order to do these tests, the transition matrix must be calculated. To obtain the necessary accuracy not obtained by numerical differencing of neighboring paths, the propagation equation (11) was integrated where the necessary partial derivative are taken to form $KH_{yy}(r)$ (see Ref. 23 for the explicit derivation) evaluated along the nominal orbit. The orbit closure is accurate to 13 places, the transition matrix was symplectic to ten significant places, and the unity eigenvalues are accurate to at least five significant places for all orbits.

The objective of the shooting method is to iteratively satisfy the boundary and transversality conditions in (6) where on each iteration (5) and (7) by virtue of (35) is satisfied. This is done in two stages by first satisfying the boundary conditions and then by traversing a one-dimensional family until the transversality condition is satisfied.

Suppose the initial value of $y(0)$ and r_p (now the stopping range since the extremal path may not be close) are guessed. Define the difference between the initial and terminal values of y as

$$\psi=y(0)-y(r_p) \tag{36}$$

A small change in ψ, $\mathrm{d}\psi$, due to a variation in $y(0)$ and r_p, is‡

$$-\mathrm{d}\psi=[\dot{y}(r_p),\Phi(r_p,0)-I]\begin{bmatrix}\mathrm{d}r\\ \delta y(0)\end{bmatrix} \tag{37}$$

Since ψ is $2n$ dimensional and $[r_p,y(0)^T]$ is $2n+1$ dimensional, some element of $[r_p,y(0)^T]$ should be specified.

‡The variation $\delta y(r)\triangleq y(r)-y^{\circ}(r)$.

Actually, two elements should be specified since in a sense one element is associated with a particular orbit and the other is associated with a particular point on the orbit. Suppose only one element is specified, say r_p. As the orbit is approached, the matrix $[\Phi(r_p,0)-I]$ becomes singular because, as noted in Sec. II, two of the eigenvalues of $\Phi(r_p,0)$ are unity for an orbit. Therfore, the numerical difficulty due to this ill-conditioned matrix as the orbit is approached is removed by fixing one element of $y(0)$. Therefore, one column of $[\Phi(r_p,0)-I]$ is removed, leaving a $2n\times 2n-1$ matrix. In general, any two elements of $[r_p,y(0)^T]$ can be fixed. By removing these two elements, the remaining $2n-1$ vector is denoted z. If the corresponding columns of $[\dot{y}(r_p),\Phi(r_p,0)-I]$ are removed, a $2n\times 2n-1$ matrix called Ω results. Therefore, Eq. (37) becomes $-\mathrm{d}\psi=\Omega\mathrm{d}z$. Since Ω is not square, it is solved in a least-square sense by taking the pseudoinverse of Ω. For this periodic cruise problem it was experienced that if the initial conditions were picked close enough to the orbit, convergence to the orbit was rapid and extremely accurate.

Once an orbit is determined, a neighboring orbit is obtained by calculating the generalized eigenvector $y_H(0)$ discussed in the section on second variational sufficient conditions for weak local optimality of period processes. This direction locally is the direction of a one-dimensional family of orbits where H is the family parameter. By making a small change along y_H and using the above closing algorithms, a neighboring orbit is obtained. By proceeding in this way, numerous points along the family are obtained and characterized by $y(0)$. This procedure is numerically inefficient. When several points along the family are found, an extrapolation routine is used to predict new neighboring orbits. The extrapolation routine chooses one of the elements of $y(0)$ as the independent variable. The other fixed element of $y(0)$ is always $\gamma(0)$. The choice of the independent variable depends upon the perceived monotonicity of that variable. If the independent variable becomes nonmonotonic, another independent variable is selected. Experience has shown that fitting a third-order curve through the last four family points works well.

Calculation of the Transition Matrix Across Thrust Switch Points

The transition matrix propagated by Eq. (11) is discontinuous at the thrust switch points. The discontinuity is due to the change in the switch ranges for an arbitrary initial perturbation in $y(0)$. The requirement is that the switch function be zero at a thrust discontinuity, i.e., $J_{r_i}=H_T(r_i)\Delta T=H_T(r_i)=0$. Therefore,

$$\mathrm{d}H_T=H_{Ty}(r_i)\,\mathrm{d}y(r_i)=0 \tag{38}$$

where d( ) denotes the total differential and $\mathrm{d}y(r_i^+)=\mathrm{d}y(r_i^-)$. The continuity of $\mathrm{d}y(r_i)$ implies that

$$\delta y(r_i^+)=\delta y(r_i^-)+[\dot{y}(r_i^-)-\dot{y}(r_i^+)]\mathrm{d}r\mathrm{d}r_i \tag{39}$$

However, from Eq. (38), using $\mathrm{d}y(r_i)=\mathrm{d}y(r_i^-)=\delta y(r_i^-)+\dot{y}(r_i^-)\,\mathrm{d}r_i$ the perturbation in $\mathrm{d}r_i$ is

$$\mathrm{d}r_i=H_{Ty}(r_i)\,\delta y(r_i^-)/H_{Ty}(r_i)\,\dot{y}(r_i^-) \tag{40}$$

Therefore, the relationship of a perturbation in δy across the thrust discontinuity is

$$\delta y(r_i^+)=\Phi(r_i^+,r_i^-)\,\delta y(r_i^-) \tag{41}$$

where

$$\Phi(r_i^+,r_i^-)=\{I+[\dot{y}(r_i^-)-\dot{y}(r_i^+)]H_{Ty}(r_i)/H_{Ty}(r_i)\dot{y}(r_i^-)\} \tag{42}$$

From the propagation equation (11) the transition matrix up to r_i^- is $\Phi(r_i^-,0)$ and using the group property of transition matrices

$$\Phi(r_i^+,0)=\Phi(r_i^+,r_i^-)\Phi(r_i^-,0) \tag{43}$$

By using $\Phi(r_i^+,0)$ to initialize (11), the transition matrix is calculated up to the next switch range.

Calculation of $J_{r_ir_i}$ and $P(r)$

Since Eq. (38) is obtained from $\mathrm{d}J_{r_i}=0$, then the determination of the variation in the switch range, as given in Eq. (40), is now shown to be of the form

$$\mathrm{d}J_{r_i}=J_{r_ir_i}\mathrm{d}r_i+J_{r_ix}\,\delta x(r_i^-) \tag{44}$$

This result occurs by noting that

$$\delta\lambda(r)=P(r)\,\delta x(r) \tag{45}$$

where $\delta\lambda(r)$ and $P(r)\delta x(r)$ satisfy the same differential equation if $P(r)$ is propagated by the Riccati differential Eqs. (12), Ref. 5. The following derivation for $J_{r_ir_i}$ is formulated so that it corresponds to that given in Refs. 16 and 17. Therefore, $J_{r_ir_i}$ is written as a function of $P(r_i^+)$ using

$$\begin{aligned}\mathrm{d}J_{r_i}&=H_{Ty}\mathrm{d}y(r_i)\Delta T=[H_{Ty}\delta y(r_i^+)+H_{Ty}\dot{y}(r_i^+)\,\mathrm{d}r_i]\Delta T\\&=\{H_{Tx}+H_{T\lambda}P(r_i^+)\}\delta x(r_i^+)+H_{Ty}\dot{y}(r_i^+)\,\mathrm{d}r_i\}\Delta T\end{aligned} \tag{46}$$

By substituting $\delta x(r_i^+)=\delta x(r_i^-)+[f(r_i^-)-f(r_i^+)]\mathrm{d}r_i$ into Eq. (46),

$$\begin{aligned}J_{r_ir_i}=\{&[H_{Tx}+H_{T\lambda}P(r_i^+)][f(r_i^-)\\&-f(r_i^+)]+H_{Ty}\dot{y}(r_i^+)\}\Delta T\end{aligned} \tag{47}$$

Noting the following relationships

$$H_{T\lambda}(r_i)=[f(r_i^-)-f(r_i^+)]/\Delta T \tag{48}$$

$$H_{Tx}(r_i)=\{\lambda^T[f_x(r_i^-)-f_x(r_i^+)]+L_x(r_i^-)-L_x(r_i^+)\}/\Delta T \tag{49}$$

$$H_{Ty}\dot{y}(r_i^+)=H_{Tx}f(r_i^+)-H_{T\lambda}[\lambda^Tf_x(r_i^+)+L_x(r_i^+)] \tag{50}$$

the curvature of the cost with respect to the switch times becomes

$$\begin{aligned}J_{r_ir_i}=&\{\lambda^T[f_x(r_i^-)-f_x(r_i^+)]+L_x(r_i^-)-L_x(r_i^+)\}f(r_i^-)\\&+[f(r_i^-)-f(r_i^+)]^TP(r_i^+)[f(r_i^-)-f(r_i^+)]\\&-[f(r_i^-)-f(r_i^+)][\lambda^Tf_x(r_i^+)+L_x(r_i^+)]\end{aligned} \tag{51}$$

This expression, also given in Refs. 16 and 17, was evaluated to determine if the switch ranges were optimizing. To evaluate Eq. (51), the Riccati solution must be calculated. First, the monodromy matrix is calculated. Then $P(0)=P(r_p)$ is detemined from Eq. (16). However, instead of integrating the Riccati equation to test for escape ranges, its solution in terms of the transition matrix given in Eq. (15) is used. Note that $P(r)$ becomes unbounded if the matrix in the indicated inversion is singular. Furthermore, note that jumps will occur in $P(r)$ at switch ranges since the transition matrix is discontinuous at these ranges.

V. Numerical Results: Periodic and Steady-State Cruise

The periodic cruise of the flat Earth atmospheric vehicle model described in Sec. III is emphasized in the following numerical results because a locally miniming orbit was determined. Furthermore, the flat Earth formulation has the advantage in that the steady-state cruise path is a dynamic extremal path satisfying the first-order necessary conditions but failing the second-order conditions for sufficiently high Mach numbers. Due to the simplicity of the modeling it is clear that fuel use is minimized by proper modulation of the potential and kinetic energy. Although the orbits for the spherical Earth had better fuel performance than the steady-state cruise, no orbit was found to be locally minimizing. Furthermore, the steady-state cruise does not satisfy even the first-order necessary conditions. It seems that one orbit that would be minimizing is the orbit about the Earth at infinite altitude.

Steady-State Cruise

The static optimization cruise problem is to minimize the integrand of (27) with respect to $\bar{h}$ and M subject to the static conditions that $M' = \gamma' = \bar{h}' = 0$. These static conditions require that

$$T = D_{ss} = qC_{D_0}S + \frac{KW^2}{qS}\left(1 - \frac{M^2a^2}{Fg(\bar{R}_o + \bar{h})}\right)^2$$

$$L_{ss} = W - \frac{M^2a^2W}{Fg(\bar{R}_o + \bar{h})}, \qquad \gamma = 0 \tag{52}$$

where D_{ss} and L_{ss} denote the drag and lift forces along the steady-state cruise path, respectively. Note that for this model the integrand of (27) reduces to $\nu D_{ss}/a$. For the spherical Earth, steady-state cruise is determined by specifying M and minimizing D_{ss} with respect to $\bar{h}$ (see Fig. 8, which shows steady-state cruise performance vs specific energy, which is defined as the total energy normalized by the vehicle weight). For the flat Earth assumption where R_o is assumed infinite, the optimal altitude and Mach number are found to be any point on a constant dynamic pressure line defined by $q^* = (K/C_{D_0})^{1/2}(W/S)$.

Along the flat Earth steady-state cruise path, the Lagrange multiplier equations reduce to $\lambda_{\bar{h}}' = \lambda_M' = \lambda_\gamma' = 0$. By using these equilibrium equations resulting from (30) to (32), the optimality condition (34), $H = 0$, $H_T = 0$, and (52), the steady-state values of $\lambda_{\bar{h}}$, λ_M, and λ_γ are obtained (see Ref. 12). Furthermore, for $T = \min_q D_{ss}$, $H_T = 0$ along the steady-state cruise path implies that the steady-state cruise path is a singular arc as was described in the Introduction. This particular atmospheric vehicle model was analyzed by a frequency type second variation in Sec. VI of Ref. 12 to determine if the steady-state extremal cruise is dynamically locally minimizing. It is shown in Ref. 12 that if

$$M > (L/D)_{\max}/2a(2\beta/g)^{1/2} \tag{53}$$

then the steady-state extremal cruise path is not locally minimizing. The main numerical result is the determination of an orbit that is locally minimizing. One important aspect of this modeling is that chattering steady-state cruise does not exist and therefore the mechanism that causes chattering is not present.

The values of the parameters used in this study are $C_{D_0} = 0.02$, $K = 0.8$, $F = 5\times10^5$, $C_L = 0.158$, $S = 24^2\,\text{ft}^2$, $g = 32.174$ ft/s², $W = 70{,}000$ lb, $\nu = 0.1\ \text{s}^{-1}$, $a = 967.705$ ft/s, $\rho_o = 4.0009819\times10^{-3}$ slugs/ft³, $\beta = 4.8100264\times10^{-5}\ \text{ft}^{-1}$, $T_{\max} = 50{,}000$ lb, $T_{\min} = 0$, $R_o = 2.1\times10^7$ ft.

The flat Earth static cruise cost is $J_{ss} = 1.83$ lb/ft. The steady-state cruise is no longer minimizing by Eq. (53) when $M > 1.24$.

Periodic Cruise Paths: Flat and Spherical Earth

A numerical program based upon the considerations described in Sec. IV was built and successfully executed. After some experimentation with initial conditions, the program converged to an orbit. Starting with this orbit, a family of orbits is generated and is represented in Fig. 1 for the flat Earth model. The Mach number associated with each family member occurs at the point on the orbit where the altitude is at a maximum and the flight path angle is zero. Both the performance index and the Hamiltonian are plotted. Note that the performance index plot for the orbit begins near the steady-state value at high Mach number and steadily decreases with M until it reaches a cusp at $M = 2.50$. Continuing from the cusp as M increases, the performance index is remarkably flat. This indicates the difficulty encountered in the convengence of first-order numerical methods to a local minimum. The two points where the Hamiltonian crosses the performance index curve represent the two orbits that satisfy the transversality condition $H = J$. However, only the monodromy matrix of the orbit at $M = 6.36$ has its four nonunity eigenvalues off the unit circle and distinct, indicating the existence of a real valued P matrix. Figure 2 shows the crossing of the Hamiltonian and performance index curves at $M = 6.36$ in more detail. The Legendre-Clebsch condition, $H_{C_LC_L} > 0$, is satisfied since λ_M varies between -2.30 and -2.75 over the orbit.

To ensure that this periodic path is locally minimizing, the $P(0)$ which generates periodic solutions to the Riccati equation is evaluated using Eq. (16). To determine whether $P(r)$ remains bounded for $r \in [0, r_p^o]$, Eq. (15) is evaluated over the orbit and is found to be bounded everywhere. Finally, Eq. (51) was evaluated at the thrust switch points and found to be positive. Thus it is concluded that the orbit at $M = 6.36$ is a locally minimizing orbit. Fuel expended using this orbit is 4.2% less than that expended on the steady-state cruise path. Furthermore, even though the steady-state path is nonunique,

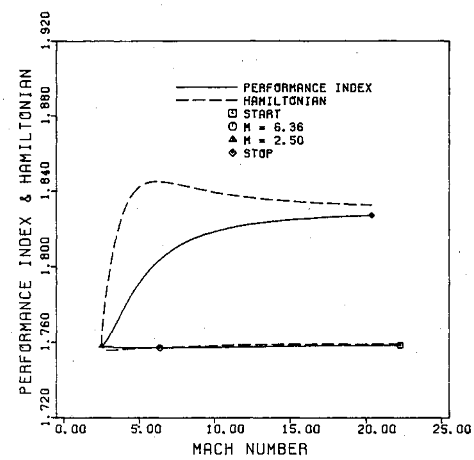

Fig. 1 Periodic family, flat Earth.

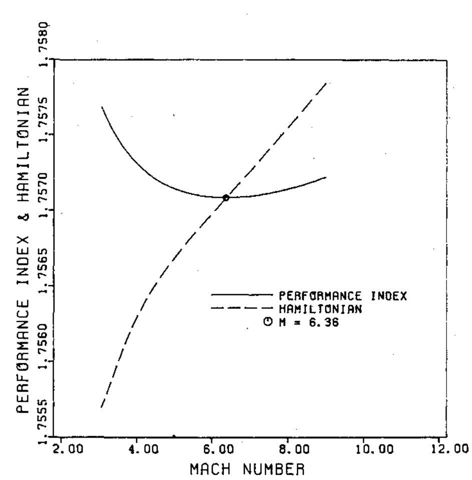

Fig. 2 Periodic family, flat Earth.

being at any (h,M) point on a dynamic pressure curve, the minimizing orbit is unique although the question of global optimality is still open.

The details of this locally minimizing orbit are shown in Figs. 3 to 6. In Fig. 3 the variation of the altitude and Mach number are shown. The discontinuity of the slopes denote the thrust switch points. Note that there are only two thrust intervals although the number of thrust switches is not limited. Clearly, along the $T_{\max}$ arc energy is increasing. After the switch point there is a potential-kinetic energy interchange culminating at the peak altitude. The Mach number at peak altitude ($\gamma=0$) is used in defining the family member in Figs. 1 and 2. The variation in flight path angle with Mach number is shown in Fig. 4. The behavior of C_L, plotted in Fig. 5 as a function of range, is quite interesting. Note that the periodic C_L makes *two* oscillations about the steady state cruise value of $C_L=0.158$. Finally, in Fig. 6 the total *g* force experienced by the vehicle is shown indicating that the maximum *g* force is only about 3. In Fig. 6, *g* force is plotted as a function of time. Figures 5 and 6, therefore, indicate the range and time intervals of the various thrusting arcs as well as for the entire orbit. It is expected that if the maximum thrust were a function of altitude, then the thrusting arcs would occur at a lower altitude on the orbit.

For the vehicle model chosen, the results for the spherical Earth are less conclusive. Figure 7 shows again the performance index and Hamiltonian as a function of Mach number for a family of orbits. An interesting aspect of this family is that the family bends back at around $M=8$. The Hamiltonian intersects the performance index at the points $M=3.77$ and $M=2.58$. Two of the nonunity eigenvalues of the monodromy matrix for the orbit characterized by $M=3.77$ lie on the unit circle and are distinct. Therefore, a real-valued, periodic solution to the Riccati equation does not exist. However, all the nonunity eigenvalues of the monodromy matrix associated with the $M=2.58$ orbit lie off the unit circle.

To determine if this periodic path is locally minimizing, the periodic generator $P(0)$ is found by satisfying Eq. (16). However, by using Eq. (15), it is found that $P(r)$ does not remain bounded for all values of $r\epsilon[0,r_p^o]$. Furthermore, Eq. (51) was evaluated at the thrust switch points and found to be negative. The conclusion is that this orbit is not locally minimizing. Nevertheless, the fuel performance of this orbit is superior to that of the associated steady-state cruise path. The steady-state cruise and periodic family performance are plotted with respect to the minimum specific energy (the lowest specific energy on the orbit) in Fig. 8. The $M=2.58$ has the best performance having decreased fuel expenditure by 4.5% from that of the steady-state cruise path at the same specific energy.

There are numerous other families of periodic paths beside those shown for both the flat and spherical Earths. The

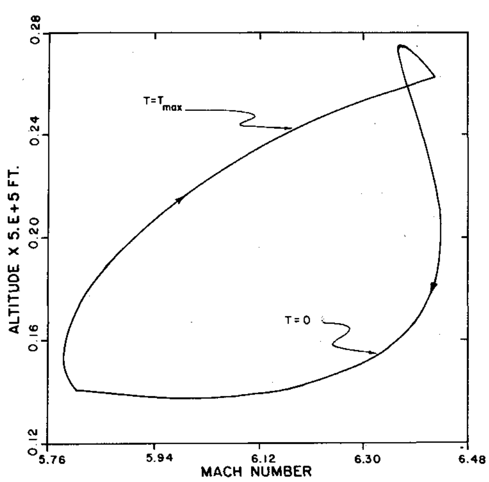

Fig. 3 Altitude vs Mach number for locally minimizing orbit (flat Earth).

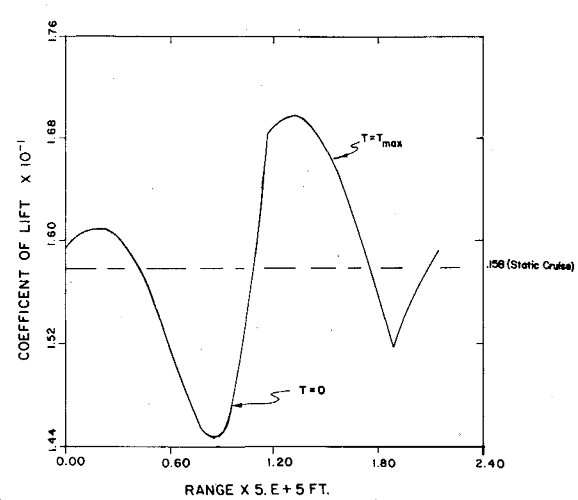

Fig. 5 C_L vs range for locally minimizing orbit (flat Earth).

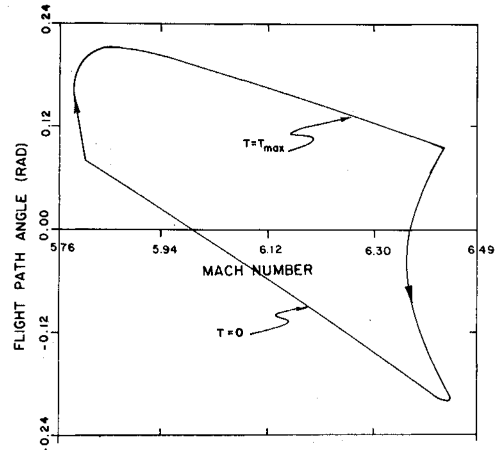

Fig. 4 Flight path angle vs Mach number for locally minimizing orbit (flat Earth).

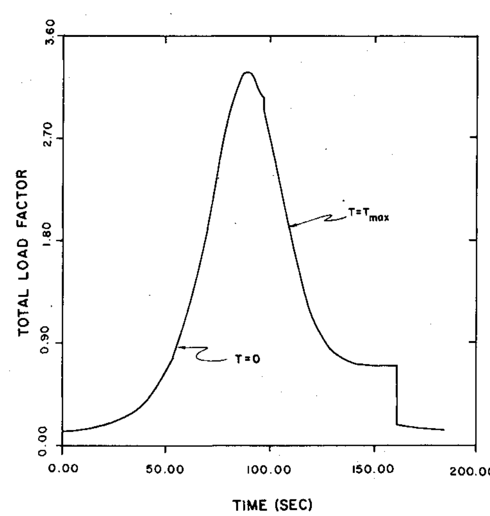

Fig. 6 *g*'s vs time for locally minimizing orbit (flat Earth).

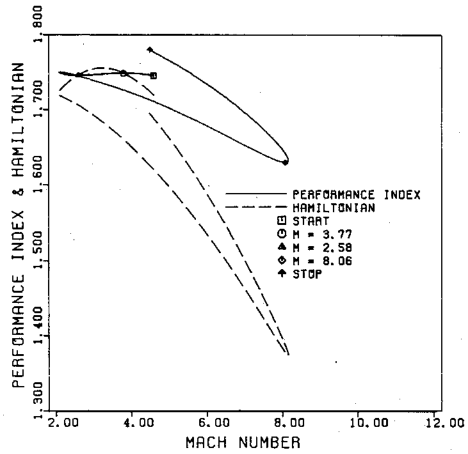

Fig. 7 Periodic family, spherical Earth.

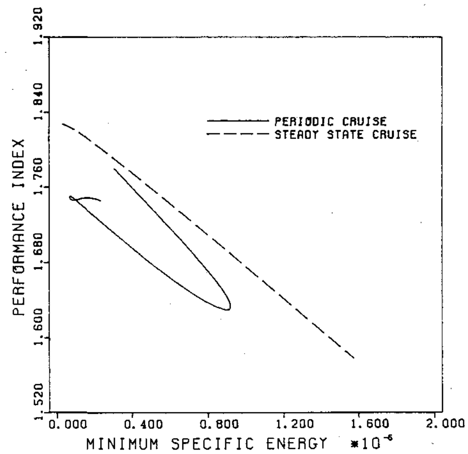

Fig. 8 Periodic vs steady-state, spherical Earth.

portion of the family where the eigenvalues of the monodromy matrix lie on the unit circle are possible bifurcation (trifurcation, etc.) points (for more detail, see Ref. 15). For example, if two of the nonunity eigenvalues are -1, then if the orbit is traced twice, the resulting monodromy matrix (for two excursions) will have four eigenvalues at unity. The rank of this monodromy matrix will be four or less. If the rank is four, then this is the bifurcation point beween a two-period family and the single-period family already exhibited. This two-period family was found in Ref. 23 and was partially traced. Its fuel performance is inferior to the present single-period family.

The difficulty with finding a weak local minimum for the spherical case might be due to competition with a trajectory which orbits the Earth such that the fuel cost is zero (the global minimum). More realistic modeling will result in the determination of a weak local minimum for the spherical Earth which occurs at suborbital speeds. Furthermore, there may exist weak local minima on some other undiscovered family of orbits.

VI. Conclusions

A point mass model of an atmospheric vehicle operating in the hypersonic region is used to investigate the fuel improvement from the steady-state cruise path obtained by modulating the flight path. For the model chosen the chattering steady-state cruise using energy-state analysis does not exist, and, therefore, the mechanism that causes fuel improvement by chattering is not present. The fuel improvement obtained is due solely to a potential-kinetic energy interchange which was indicated by a frequency type second variational analysis of the steady-state cruise for the flat Earth model. By developing a shooting method tailored to the peculiarities associated with periodic paths of Hamiltonian systems, a family of orbits is generated for both the flat and spherical Earth models. By applying the second variational sufficiency conditions for periodic processes, only one orbit which involves the flat Earth model is found to be locally minimizing. The existence and determination of this orbit is one of the unique contributions of this work.

Acknowledgment

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